

KFKI-1985-28
HU ISSN 0368 5330
ISBN 963 372 361 2
March 1985

ORTHOGONAL JUMPS OF WAVEFUNCTION IN WHITE-NOISE POTENTIALS

L. Diósi

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

ABSTRACT

We investigate the evolution of the quantum state for a free particle placed into a random external potential of white-noise type. The master equation for the density matrix is derived by means of path integral method. We propose an equivalent stochastic process where the wavefunction satisfies a nonlinear Schrödinger equation except for random moments at which it shows orthogonal jumps. The relation of our work to the usual theory of quantum noise and damping is briefly discussed.

Since the early works [1,2,3] on the theory of quantum noise and damping, a great interest has been shown in quantummechanical systems affected by random forces and also, new aspects have appeared [4,5,10]. Here we shall investigate the effect of Gaussian white-noise potentials.

For brevity we restrict ourselves to the case of a single point-line particle moving in one dimension.

Let us assume that the potential $V(x, t)$ acting on the particle is a stochastic variable (x, t stand for the coordinate and time). We define the probability distribution of V by the following generator functional $G[h]$:

$$\begin{aligned} G[h] &\equiv \langle \exp(i \int V(r, t) h(r, t) dr dt) \rangle = \\ &= \exp(-\frac{1}{2} \int h(r, t) h(r', t) f(r - r') dr dr') \end{aligned} \quad (1)$$

where h is an arbitrary function. The symbol $\langle \rangle$ stands for expectation values evaluated by means of the probability distribution of V . Functional differentiation of $G[h]$ gives rise to the two typical relations of moments of white-noise type:

$$\langle V(r, t) \rangle = 0, \quad (2)$$

$$\langle V(r, t) V(r', t') \rangle = \delta(t - t') f(r - r'). \quad (3)$$

For further use, we introduce the following notation:

$$g(r) = f(0) - f(r). \quad (4)$$

Now we investigate the effect of the white-noise potential (1) on the quantummechanical motion of a given point-like particle of mass m .

If we single out a given potential V then the wavefunction $\psi_t(x)$ of the particle will satisfy the Schrödinger equation of motion:

$$\frac{\partial}{\partial t} \psi_t(x) = \frac{i\hbar}{2m} \frac{\partial^2}{\partial x^2} \psi_t(x) - \frac{i}{\hbar} V(x, t) \psi_t(x) \quad (5)$$

where x is the spatial coordinate and t refers to the actual value of the time. Taking the initial wavefunction $\psi_0(x)$ at $t = 0$, one can express the solution for $\psi_t(x)$ by means of the Feynman's path integral formula [7]:

$$\psi_t(x) = \int \exp\left\{ \frac{i}{\hbar} \int_0^t \left[\frac{m}{2} \dot{x}_\tau^2 - V(x_\tau, \tau) \right] d\tau \right\} \psi_0(x_0) Dx_\tau. \quad (6)$$

$t > \tau \geq 0$

In our case, $V(x, t)$ is a stochastic variable, thus $\psi_t(x)$ will evolve in time according to a give stochastic process. We shall not derive the rules of this stochastic process. Instead, we shall construct another stochastic process for $\psi_t(x)$ which leads to the correct statistical predictions and which has genuine features from the viewpoint of measurement theory.

In the generic case, the quantum state of the particle is uniquely characterized by the density matrix [6]

$$\rho_t(x, y) \equiv \langle \psi_t(x) \psi_t^*(y) \rangle. \quad (7)$$

Indeed, it can be shown [8] that ρ_t yields all the usual statistical predictions of the quantummechanics. Namely,

$$O_t = \int \hat{O}(y, x) \rho_t(x, y) dx dy \quad (8)$$

where \hat{O} is the hermititian operator of an arbitrarily given dynamical quantity and O_t stands for its predicted value at time t .

First, we shall prove that the density matrix (7) satisfies a parabolic differential equation of motion. Using eq. (6) along with eq. (7), one gets:

$$\begin{aligned} \rho_t(x, y) = & \\ = \langle \int \exp\{ \frac{i}{\hbar} \int_0^t [\frac{m}{2} (\dot{x}_\tau^2 - \dot{y}_\tau^2) - (V(x_\tau, \tau) - V(y_\tau, \tau))] d\tau \} \rho_0(x_0, y_0) Dx_\tau Dy_\tau \rangle & \\ & t > \tau \geq 0 \end{aligned} \quad (9)$$

where ρ_0 is the initially given density matrix: $\rho_0(x, y) = \psi_0(x) \psi_0^*(y)$. On the rhs one can substitute

$$\langle \exp\{ -\frac{i}{\hbar} \int_0^t [V(x_\tau, \tau) - V(y_\tau, \tau)] d\tau \} \rangle = \exp\{ -\frac{1}{\hbar^2} \int_0^t g(x_\tau - y_\tau) d\tau \}, \quad (10)$$

which obviously follows from eq. (1) if we insert there $h(r, t) = -(1/\hbar)(\delta(r - x_\tau) - \delta(r - y_\tau))$ and use eq. (4). Thus we have path integral representation for the density matrix at arbitrary time t :

$$\begin{aligned}
\rho_t(x, y) &= \\
&= \int_0^t \exp\left\{ \int_0^\tau \left[i \frac{m}{2\hbar} (\dot{x}_\tau^2 - \dot{y}_\tau^2) - \frac{1}{\hbar^2} g(x_\tau - y_\tau) \right] d\tau \right\} \rho_0(x_0, y_0) Dx_\tau Dy_\tau. \\
&\hspace{15em} t > \tau \geq 0
\end{aligned} \tag{11}$$

Differentiating both sides of eq. (11) by t , one gets the equation of motion for the density matrix in Gaussian white-noise potential (1):

$$\frac{\partial}{\partial t} \rho_t(x, y) = \frac{i\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \rho_t(x, y) - \frac{1}{\hbar^2} g(x - y) \rho_t(x, y). \tag{12}$$

This equation is sometimes called the master equation of the quantum noise theory. Our path integral method seems to be very effective for deriving the master equation even in more general external noises.

The second term on the rhs of eq. (12) is a typical damping term known from the quantum theory of reservoir effects (c.f. coarse grained approximation in ref. 3). This term cannot be reproduced by any given stochastic hamiltonian.

Beside damping, a very peculiar feature of eq. (12) is that it produces mixed quantum state from a pure one in a continuous manner. Exploiting the nature of this permanent quantum state mixing we shall construct the stochastic process for the evolution of the wavefunction itself.

In order to make the calculation as simple as possible we suppose that

$$g(r) = \frac{1}{2} A^2 r^2 + \text{higher order terms in } r; \quad A = \text{const.} \tag{13}$$

and we shall neglect the “higher terms” by assuming that the width of the wavefunction will always be small enough.

Thus, eq. (12) takes the form

$$\frac{\partial}{\partial t} \rho_t(x, y) = \left[\frac{i\hbar}{2m} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} \right) \rho_t(x, y) - \frac{A^2}{2\hbar^2} (x - y)^2 \right] \rho_t(x, y). \tag{14}$$

If at time t the particle is in a pure quantum state with the given wavefunction ψ_t then [6]

$$\rho_t(x, y) = \psi_t(x) \psi_t^*(y) \tag{15}$$

and eq. (14) yields

$$\begin{aligned} \rho_{t+\epsilon}(x, y) &= \\ &= \left[1 - \frac{\epsilon A^2}{2\hbar^2}(x - y)^2\right] \left[1 + i\frac{\hbar\epsilon}{2m}\frac{\partial^2}{\partial x^2}\right] \psi_t(x) \left[1 - i\frac{\hbar\epsilon}{2m}\frac{\partial^2}{\partial y^2}\right] \psi_t^*(y) \end{aligned} \quad (16)$$

for infinitesimal $\epsilon > 0$. Now, the rhs is not a single product like it was in eq. (15). Nevertheless, it can be decomposed into the sum of two such diadic terms:

$$\rho_{t+\epsilon}(x, y) = (1 - \epsilon w)\psi_{t+\epsilon}(x)\psi_{t+\epsilon}^*(y) + \epsilon w\psi_{t+\epsilon}(x)\tilde{\psi}_{t+\epsilon}^*(y) \quad (17)$$

where

$$w = \frac{A^2}{\hbar^2}\sigma_\psi^2 \quad (18)$$

is the mixing rate, the dominant wavefunction $\psi_{t+\epsilon}$ is

$$\psi_{t+\epsilon}(x) = \left\{1 - \frac{\epsilon A^2}{2\hbar^2}[(x - x_\psi)^2 - \sigma_\psi^2] + i\frac{\epsilon\hbar}{2m}\frac{\partial^2}{\partial x^2}\right\}\psi_t(x), \quad (19)$$

and the contaminating wavefunction $\tilde{\psi}_{t+\epsilon}$ is

$$\tilde{\psi}_{t+\epsilon}(x) = \left(\frac{x - x_\psi}{\sigma_\psi} + \frac{\epsilon A^2}{2\hbar^2}\frac{a_\psi^3}{\sigma_\psi}\right)\psi_t(x). \quad (20)$$

We introduced the following notations:

$$\begin{aligned} x_\psi &= \int x|\psi_t(x)|^2 dx, \\ \sigma_\psi^2 &= \int (x - x_\psi)^2 |\psi_t(x)|^2 dx, \\ a_\psi^3 &= \int (x - x_\psi)^3 |\psi_t(x)|^2 dx. \end{aligned} \quad (21)$$

It is easy to verify that $\psi_{t+\epsilon}$, $\tilde{\psi}_{t+\epsilon}(x)$ are normalized to the unity and orthogonal to each other (in the lowest order of ϵ , of course).

Now, let us read out the statistical meaning of eq. (17): in an infinitesimally short time ϵ , the quantum state ψ_t of the particle either evolves continuously into the neighbouring state $\psi_{t+\epsilon}$, or, with the infinitesimal probability ϵw , jumps to the state $\tilde{\psi}_{t+\epsilon}$, which is orthogonal to $\psi_{t+\epsilon}$.

Thus, the stochastic process governing the evolution of the wavefunction is as follows. The wavefunction $\psi_t(x)$ of the given particle satisfies the following non-linear equation of motion [9]:

$$\begin{aligned}\frac{\partial}{\partial t}\psi_t(x) &= i\frac{\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi_t(x) - \frac{A^2}{2\hbar^2}[(x - x_\psi)^2 - \sigma_\psi^2]\psi_t(x); \\ x_\psi &= \int x|\psi_t(x)|^2 dx, \\ \sigma_\psi^2 &= \int (x - x_\psi)^2 |\psi_t(x)|^2 dx,\end{aligned}\tag{22}$$

apart from discrete orthogonal jumps

$$\psi_{t+0} = \frac{x - x_\psi}{\sigma_\psi}\psi_t(x)\tag{23}$$

which occur at random in time. A jump is performed in an infinitesimal interval $(t, t + \epsilon)$ with probability

$$w\epsilon = \frac{A^2}{\hbar^2}\sigma_\psi^2\epsilon.\tag{24}$$

By its construction, this stochastic process leads to the same physical predictions in average as eq. (8) did. Namely, given the initial density matrix $\rho_0(x, y)$, we can decompose it as

$$\rho_0(x, y) = \sum_r p_r \psi_0^{(r)} \psi_0^{*(r)}(y)\tag{25}$$

where $\psi_0^{(r)}$'s form an orthonormal system. Let us regard equality (25) as if the particle were in the pure state $\psi_0^{(r)}$ with probability p_r . Starting the stochastic process (22,23,24) from these initial wavefunctions, each of them will give rise to the quantity

$$\sum_r p_r \psi_t^{(r)} \psi_t^{*(r)}(y).\tag{26}$$

The stochastic average of this expression over the histories $\psi_t^{(r)}$ is equal to $\rho_t(x, y)$.

We have to note that many other stochastic processes for ψ_t can be constructed with the same $\rho_t(x, y)$. Nevertheless, we would like to underline that

the orthogonality of the stochastic jumps (23) is very crucial from the viewpoint of measurement theory: It can be shown that if we know the wavefunction ψ_t at $t = 0$ then, by means of von Neumann measurements [8], we can register all stochastic jumps, without disturbing the measured particle. Thus, in every moment, we are able to find out the wavefunction of the system, if it satisfies indeed the eqs. (22,23,24). Of course, it is not obvious at all how should we realize the proper measuring apparatuses in practice.

Finally, we note that in the Langevin approach of quantum damping [2] also stochastic process is constructed but this process is related to the evolution of the density matrix, not to the pure quantum state of the system.

I wish to thank to the authors of ref. 4 and also to Dr. P. Hraskó for illuminating discussions.

REFERENCES

- [1] W.H. Louisell and L.R. Walker, Phys. Rev. 137, B204 (1965)
- [2] M. Lax, Phys. Rev. 145, 110 (1966)
- [3] W.H. Louisell, Quantum Statistical Properties of Radiation (Wiley, New York, 1973)
- [4] F. Károlyházi, A. Frenkel and B. Lukács, in: Physics as Natural Philosophy, eds. A. Shimony and H. Feshbach (MIT Press, Cambridge, 1982)
- [5] J.M. Ziman, Models of Disorder (Cambridge University Press, Cambridge, 1977)
- [6] L.D. Landau and E.M. Lifshits, Quantum Mechanics (Pergamon, Oxford, 1977)
- [7] R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals (McGraw-Hill Book Company, New York, 1965)
- [8] J. von Neumann, Mathematische Grundlagen der Quantenmechanik (Springer Verlag, Berlin, 1932)
- [9] The equation (22) in itself, possesses solitonlike solutions. For similar mechanism see, e.g., L. Diósi, Phys. Lett. 105A, 199 (1984);. I. Bialynicki-Birula and J. Mycielski: Ann. Phys. 100, 62 (1976)
- [10] N. Gisin, Phys. Rev. A28, 2891 (1983)